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On the Regularity of Parabolic Equations and Obstacle Problems with Quadratic Growth Nonlinearities

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1. INTRODUCTION

In this paper we study regularity of second order quasilinear parabolic equations and obstacle problems with quadratic growth nonlinearities

$$u_t - (a_i(x, t, u, \nabla u))_{x_i} = b(x, t, u, \nabla u).$$

We prove Hölder continuity of the bounded weak solution and Hölder continuity of the gradient of the weak solution in the interior by using a perturbation method similar to [3].

In the stationary case J. Frehse and U. Mosco [6] considered a large class of V.I. and proved $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity for weak solutions. Also $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity for a more general class of elliptic obstacle problems have been studied by a number of authors.

For the evolutionary case M. Struwe and M. A. Vivaldi [14] showed Hölder continuity of the weak solution when the obstacle is Hölder continuous. Indeed they used Green's function and hole filling technique to estimate a Morrey type growth condition. Also M. Biroli [1] showed the existence of a Hölder continuous solution to parabolic V.I. with quadratic growth nonlinearities under the assumption that the obstacle is in $C^{\alpha,\alpha/2}$. In fact Hölder continuity of the weak solution has been studied by a number of authors mainly under the assumption that the obstacle is Hölder continuous.

We show Hölder continuity of the weak solution under the assumption that the x derivative of the obstacle is in L^{q_1}, $q_1 > N$, for each t and the t derivative is in L^{q_2}, $q_2 > (N+2)/2$, where N is the dimension of space. Here we do not assume that the obstacle is continuous in time. For the proof we show a reverse Hölder inequality which is essential for the perturbation argument. Then by a careful comparison argument we show a Campanato type growth condition for the solution. Consequently the Hölder continuity follows from a known iteration lemma.

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As far as we know, this seems to be the first paper which proves Hölder continuity of the gradient of solutions for parabolic variational inequalities with quadratic growth. We assume that the gradient of the obstacle is Hölder continuous and the time derivative is in a sufficiently high L^p space. Since Hölder continuity of a solution can be proved with a large Hölder exponent, we can use a perturbation argument in a relatively simple way. In fact we show that the gradient of solution satisfies a Campanato type growth condition.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, T a positive number, and $Q = \Omega \times (0, T)$. Let $a_i(x, t, u, h)$; $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be measurable in x and t, continuous in (u, h), and let $a_i(x, t, u, h)$ be differentiable in h. We assume the growth conditions

$$|a_i(x, t, u, h)| \leqslant c(|h| + 1), \qquad \left|\frac{\partial a_i}{\partial h_j}\right| \leqslant c$$

and the ellipticity condition

$$\frac{\partial a_i}{\partial h_j} \xi_i \xi_j \geqslant c\, |\xi|^2, \qquad \text{for all } \xi \in \mathbb{R}^N \text{ and for some } c > 0,$$

where the double indices mean summation up to N and c depends on $M = \sup |u|$. Also we assume $b(x, t, u, h)$ is a Carathéodory function satisfying a quadratic growth condition

$$|b(x, t, u, h)| \leqslant k_1(1 + |h|^2)$$

for a certain nonnegative constant k_1 depending on M.

Suppose $u \in K = \{v \in L^2(0, T\colon W_0^{1,2}(\Omega)) \cap L_\infty(Q),\ v \geqslant \psi\}$ satisfies

$$\int_0^T \int_\Omega v_t(v - u) + a_i(x, t, u, \nabla u)(v - u)_{x_i} - b(x, t, u, \nabla u)(v - u)\, dx\, dt$$
$$+ \tfrac{1}{2} \|v(0)\|^2_{L^2(\Omega)} \geqslant 0 \tag{1}$$

for all $v \in K$ where ψ is the obstacle. We assume

$$\|u\|_{L^\infty(Q)} \leqslant M.$$

We define

$$B_R = \{x \in \mathbb{R}^N\colon |x - x_0| < R\},$$
$$Q_R = B_R \times (t_0 - R^2, t_0),$$
$$Q_{R,\varepsilon} = B_R \times (t_0 - \varepsilon R^2, t_0),$$

$$\Lambda_{R,\varepsilon} = [t_0 - \varepsilon R^2, t_0],$$
$$\Lambda'_{R,\varepsilon} = [t_0 - 2\varepsilon R^2, t_0 - \varepsilon R^2],$$
$$\partial_p Q_R = \{(x, t) \in \bar{Q}_R : t = t_0 - \varepsilon R \text{ or } |x - x_0| = R^2\}.$$

We take $\eta \in C_0^\infty(B_{2R})$, $\eta \equiv 1$ in B_R, $0 \leqslant \eta \leqslant 1$, and $|\nabla\eta| \leqslant 5/R$. We take $\tau \in C^\infty(\mathbb{R})$, $0 \leqslant \tau \leqslant 1$, $\tau(t) \equiv 0$ for $t \leqslant -4R^2$, $\tau(t) \equiv 1$ for $t \geqslant -R^2$, and $|\tau_t| \leqslant 5/R^2$. We define $\chi_{[a,b]}$ to be the characteristic function such that $\chi_{[a,b]} = 1$ for $a \leqslant t \leqslant b$ and $\chi_{[a,b]}(t) = 0$ otherwise.

We define $z = (x, t)$. We assume the generic point $(x_0, t_0) = (0, 0)$ if there is no confusion. We denote c by a universal constant depending only on data.

Remark 0.1. We consider only obstacle problems. If we put $\psi = -M$, then all the proofs work for equations.

2. A Reverse Hölder Inequality

In this section we assume that

$$\psi \in W^{1,q}(Q) \cap L^\infty(Q), \qquad q > 2.$$

We denote

$$\Psi_{2R,\varepsilon} = \sup_{t \in \Lambda_{2R,\varepsilon}} \int_{B_{2R}} |\psi - \tilde{\psi}_{2R,t}|^2 \, dx + \int_{Q_{2R,\varepsilon}} |\nabla\psi|^2 + R^2 \, |\psi_t|^2 + 1 \, dz,$$

where we defined

$$\tilde{\psi}_{2R,t} = \frac{1}{\int_{B_{2R}} \eta^2 \, dx} \int_{B_{2R}} \psi(x, t) \, \eta^2 \, dx.$$

Since we can write

$$a_i(z, u, h) - a_i(z, u, 0) = \int_0^1 a_{i,h_j}(z, u, sh) \, ds \, h_j$$

for all (z, u) by the mean value theorem, we need only to consider the case

$$a_i(z, u, h) = a_{ij}(z) \, h_j$$

in Sections 2 and 3.

First we prove a lemma which is essential for a Poincaré inequality for the solutions of parabolic equations. A similar lemma can be found in [13] or [14].

LEMMA 1. *There exists a small* ε_0 *depending only on* M, N *such that for each* $0<\varepsilon<\varepsilon_0$

$$\begin{aligned}
\sup_{t\in\Lambda_{R,\varepsilon}} \int_{\Lambda'_{R,\varepsilon}} ds \int_{B_{2R}} \eta^2\, |u(x,t)-\tilde{u}_{2R,s}|^2\, dx
&\leqslant \delta\varepsilon\, R^2 \int_{Q_{2R,\varepsilon}} |\nabla u|^2\, dz \\
&\quad + c\int_{\Lambda'_{R,\varepsilon}} ds \int_{B_{2R}} |u(x,s)-\tilde{u}_{2R,s}|^2\, \eta^2\, dx + cR^2\Psi_{2R,\varepsilon}
\end{aligned} \tag{2}$$

for some small δ.

Proof. For any $s\in\Lambda'_\varepsilon$ consider $\bar{t}=\bar{t}(s)\geqslant s$ such that

$$\int_{B_{2R}} \eta^2\, |u(x,\bar{t})-\tilde{u}_{2R,s}|^2\, dx = \sup_{0\geqslant t\geqslant s} \int_{B_{2R}} \eta^2\, |u(x,t)-\tilde{u}_{2R,s}|^2\, dx. \tag{3}$$

The existence of $\bar{t}$ can be found in [14]. Let $v=u-\rho(\bar{u}-\bar{\psi})\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}$; then $v\in K$ if ρ is small enough, where $\bar{u}=u-\tilde{u}_{2R,s}$ and $\bar{\psi}=\psi-\tilde{\psi}_{2R,s}$. So taking v as a competing function to (1) we have

$$\begin{aligned}
&\int (\bar{u}-\bar{\psi})_t\, (\bar{u}-\bar{\psi})\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\, dz \\
&\qquad + \int |\nabla\bar{u}-\nabla\bar{\psi}|^2\, \{1+2\mu\, |\bar{u}-\bar{\psi}|^2\}\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\, dz \\
&\leqslant \int |\bar{\psi}_t|\, |\bar{u}-\bar{\psi}|\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\, dz \\
&\qquad + c\int |\nabla\bar{u}-\nabla\bar{\psi}|\, |\nabla\bar{\psi}|(1+2\mu\, |\bar{u}-\bar{\psi}|^2)\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\, dz \\
&\qquad + c\int |\nabla\bar{\psi}|\, |\nabla\eta|\, \eta\, |\bar{u}-\bar{\psi}|\, e^{\mu(\bar{u}-\bar{\psi})^2}\chi_{[s,\bar{t}]}\, dz \\
&\qquad + c\int |\nabla\bar{u}-\nabla\bar{\psi}|\, |\nabla\eta|\, \eta\, |\bar{u}-\bar{\psi}|\, e^{\mu(\bar{u}-\bar{\psi})^2}\chi_{[s,\bar{t}]}\, dz \\
&\qquad + c\int (|\nabla\bar{u}|^2+1)(|\bar{u}-\bar{\psi}|)\, e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\, dz.
\end{aligned} \tag{4}$$

Since the first term in (4) can be written

$$\int (\bar{u}-\bar{\psi})_t\,(\bar{u}-\bar{\psi})\,e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\,dz$$

$$=\frac{1}{2\mu}\int (e^{\mu(\bar{u}-\bar{\psi})^2}-1)\,\eta^2(x,\bar{t})\,dx-\frac{1}{2\mu}\int (e^{\mu(\bar{u}-\bar{\psi})^2}-1)\,\eta^2(x,s)\,dx,$$

we have for sufficiently large μ

$$\frac{1}{\mu}\int (e^{\mu(\bar{u}-\bar{\psi})^2}-1)\,\eta^2(x,\bar{t})\,dx$$

$$+\int |\nabla\bar{u}|^2\,(1+|\bar{u}-\bar{\psi}|^2)\,e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\,dz$$

$$\leqslant c\int (|\nabla\bar{\psi}|^2+R^2\,|\bar{\psi}_t|^2+1)\,\chi_{[s,\bar{t}]}\,dz$$

$$+\delta\int_{Q_{2R,\varepsilon}} |\nabla\bar{u}|^2\,dz+\frac{c}{R^2}\int (|\bar{u}|^2+|\bar{\psi}|^2)\,\eta^2\chi_{[s,\bar{t}]}\,dz$$

$$+c\int (e^{\mu(\bar{u}-\bar{\psi})^2}-1)\,\eta^2(x,s)\,dx, \tag{5}$$

where we used Young's inequality several times.

Since u and ψ are bounded, we see that for some c

$$\mu(\bar{u}-\bar{\psi})^2\leqslant e^{\mu(\bar{u}-\bar{\psi})^2}-1\leqslant c(\bar{u}-\bar{\psi})^2.$$

By Poincaré's inequality we have

$$\frac{1}{R^2}\int_{Q_{2R,\varepsilon}} |\bar{\psi}|^2\,dz\leqslant c\Psi_{2R,\varepsilon}.$$

Also we have by definition of $\bar{t}$

$$\frac{1}{R^2}\int |\bar{u}|^2\,\eta^2\chi_{[s,\bar{t}]}\,dz\leqslant \varepsilon\int |\bar{u}|^2\,\eta^2(x,\bar{t})\,dx.$$

So we can write (5)

$$\int |\bar{u}-\bar{\psi}|^2\,\eta^2(x,\bar{t})\,dx+\int |\nabla\bar{u}|^2\,(1+|\bar{u}-\bar{\psi}|^2)\,e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\chi_{[s,\bar{t}]}\,dz$$

$$\leqslant c\Psi_{2R,\varepsilon}+c\varepsilon\int |\bar{u}|^2\,\eta^2(x,\bar{t})\,dx$$

$$+\delta\int_{Q_{2R,\varepsilon}} |\nabla\bar{u}|^2\,dz+c\int |\bar{u}-\bar{\psi}|^2\,\eta^2(x,s)\,dx$$

and if ε is small enough that

$$\int |\bar{u}|^2 \eta^2(x, \bar{t})\, dx \leqslant c\Psi_{2R,\varepsilon} + \delta \int_{Q2R,\varepsilon} |\nabla \bar{u}|^2\, dz + c \int |\bar{u}|^2 \eta^2(x, s)\, dx. \quad (6)$$

Integrating (6) with respect to s from $-2\varepsilon R^2$ to $-\varepsilon R^2$, we prove (2). ∎

Following [14] we prove Lemma 2 which gives a Poincaré inequality for a parabolic equation with Lemma 1.

LEMMA 2. *Suppose $Q_{2R,\varepsilon} \subset Q$. Then for some small δ*

$$\int_{Q_{R,\varepsilon}} |u - u_{R,\varepsilon}|^2\, dz \leqslant \delta\varepsilon R^2 \int_{Q_{2R,\varepsilon}} |\nabla u|^2\, dz + cR^2\Psi_{2R,\varepsilon}$$

$$+ c \int_{A'_\varepsilon} ds \int_{B_{2R}} |u - \tilde{u}_{2R,s}|^2 \eta^2\, dx, \quad (7)$$

where

$$u_{R,\varepsilon} = \frac{1}{|Q_{R,\varepsilon}|} \int_{Q_{R,\varepsilon}} u(x, t)\, dz.$$

Now we prove a Cacciopoli type inequality.

LEMMA 3. *Suppose $Q_{2R,\varepsilon} \subset Q$. Then*

$$\int_{Q_{R,\varepsilon}} |\nabla u|^2\, dz \leqslant \frac{c}{\varepsilon R^2} \int_{Q_{2R,\varepsilon}} |u - u_{2R,\varepsilon}|^2\, dz + c\Psi_{2R,\varepsilon}. \quad (8)$$

Proof. Again we take $v = u - \rho(\bar{u} - \bar{\psi})\, e^{\mu(\bar{u} - \bar{\psi})^2} \eta^2 \tau_\varepsilon \chi_{[-\infty, 0]}$ as a competing function in (1), where $\tau_\varepsilon(t) = \tau(t/\varepsilon)$, $\bar{u} = u - u_{2R,\varepsilon}$ and $\bar{\psi} = u - \psi_{2R,\varepsilon}$. So from (1) we conclude

$$\frac{1}{\mu} \int \frac{\partial}{\partial t} \{(e^{\mu(\bar{u} - \bar{\psi})^2} - 1)\, \eta^2 \tau_\varepsilon\}\, \chi_{[-\infty, 0]}\, dz$$

$$+ \int |\nabla \bar{u} - \nabla \bar{\psi}|^2 (1 + \mu\, |\bar{u} - \bar{\psi}|^2)\, e^{\mu(u - \bar{\psi})^2} \eta^2 \tau_\varepsilon \chi_{[-\infty, 0]}\, dz$$

$$\leqslant c \int |\psi_t|\, |\bar{u} - \bar{\psi}|\, e^{\mu(\bar{u} - \bar{\psi})^2} \eta^2 \tau_\varepsilon \chi_{[-\infty, 0]}\, dz$$

$$+ \frac{c}{\mu} \int (e^{\mu(\bar{u} - \bar{\psi})^2} - 1)\, \eta^2\, |\tau_{\varepsilon,t}|\, \chi_{[-\infty, 0]}\, dz$$

$$+c\int(1+|\nabla\bar{u}|^2)\,|\bar{u}-\bar{\psi}|\,e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\tau_\varepsilon\chi_{[-\infty,0]}\,dz$$

$$+c\int|\nabla\bar{u}-\nabla\bar{\psi}|\,|\nabla\eta|\,\eta\,|\bar{u}-\bar{\psi}|\,e^{\mu(\bar{u}-\bar{\psi})^2}\tau_\varepsilon\chi_{[-\infty,0]}\,dz$$

$$+c\int|\nabla\psi|\,|\nabla\bar{u}-\nabla\bar{\psi}|(1+|\bar{u}-\bar{\psi}|^2)\,e^{\mu(\bar{u}-\bar{\psi})^2}\eta^2\tau_\varepsilon\chi_{[-\infty,0]}\,dz$$

$$+c\int|\nabla\bar{\psi}|\,|\nabla\eta|\,\eta\,|\bar{u}-\bar{\psi}|\,e^{\mu(\bar{u}-\bar{\psi})^2}\tau_\varepsilon\chi_{[-\infty,0]}\,dz. \tag{9}$$

By Young's inequality we have for sufficiently large μ

$$\sup_{t\in\Lambda_{R,\varepsilon}}\frac{1}{\mu}\int_{B_{2R}}(e^{\mu(\bar{u}-\bar{\psi})^2}-1)\,\eta^2\tau_\varepsilon\,dx+\int|\nabla\bar{u}|^2\,\eta^2\tau_\varepsilon\chi_{[-\infty,0]}\,dz$$

$$\leqslant c\int(R^2\,|\bar{\psi}_t|^2+|\nabla\bar{\psi}|^2)\,\eta^2\tau_\varepsilon\chi_{[-\infty,0]}\,dz$$

$$+\frac{c}{\varepsilon R^2}\int_{Q_{2R,\varepsilon}}|\bar{u}|^2+|\bar{\psi}|^2\,dz \tag{10}$$

and Lemma 3 follows immediately from (10). ∎

LEMMA 4. *Suppose $Q_{2R,\varepsilon}\subset Q$. Then we have*

$$\sup_{t\in\Lambda_{R,\varepsilon}}\int_{B_R}|u(x,t)-\tilde{u}_{R,t}|^2\,dx\leqslant c\int_{Q_{2R,\varepsilon}}|\nabla u|^2\,dz+c\Psi_{2R,\varepsilon} \tag{11}$$

for some c and small ε.

Proof. Define $\bar{u}=u-\tilde{u}_{2R,t}$ and $\bar{\psi}=\psi-\tilde{\psi}_{2R,t}$. Taking $v=u-\rho(\bar{u}-\bar{\psi})\,\eta^2\tau\chi_{[-\infty,t_1]}$, $t_1\in\Lambda_{R,\varepsilon}$ as a competing function to (1), we have

$$\int\frac{\partial}{\partial t}\{(\bar{u}-\bar{\psi})^2\,\eta^2\tau_\varepsilon\}\,\chi_{[-\infty,t_1]}\,dz+\int|\nabla\bar{u}-\nabla\bar{\psi}|^2\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$\leqslant c\int|\bar{\psi}_t|\,|\bar{u}-\bar{\psi}|\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$+c\int|\bar{u}-\bar{\psi}|^2\,\eta^2\,|\tau_{\varepsilon,t}|\,\chi_{[-\infty,t_1]}\,dz$$

$$+c\int|\nabla\bar{\psi}|\,|\nabla\eta|\,\eta\,|\bar{u}-\bar{\psi}|\,\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$+c\int |\nabla\bar{\psi}|\,|\nabla\bar{u}-\nabla\bar{\psi}|\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$+c\int |\nabla\bar{u}-\nabla\bar{\psi}|\,|\nabla\eta|\,\eta\,|\bar{u}-\bar{\psi}|\,\tau\chi_{[-\infty,t_1]}\,dz$$

$$+c\int (1+|\nabla u|^2)\,|\bar{u}-\bar{\psi}|\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz. \tag{12}$$

By Young's inequality we get

$$\int |u-\tilde{u}_{2R,t_1}|^2\,\eta^2(x,t_1)\,dx$$

$$\leqslant c\int_{Q_{2R,\varepsilon}} |\nabla u|^2\,dz+\frac{c}{R^2}\int |u-\tilde{u}_{2R,t}|^2\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$+c\int |\psi-\tilde{\psi}_{2R,t_1}|^2\,\eta^2(x,t_1)\,dx+cR^2\int |\psi_t|^2\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz$$

$$+c\int |\nabla\psi|^2\,\eta^2\tau_\varepsilon\chi_{[-\infty,t_1]}\,dz+cR^{N+2}$$

$$\leqslant c\int_{Q_{2R,\varepsilon}} |\nabla u|^2\,dz+c\Psi_{2R,\varepsilon}, \tag{13}$$

where we used a kind of Poincaré inequality

$$\frac{c}{R^2}\int |u-\tilde{u}_{2R,t}|^2\,\eta^2\tau_\varepsilon\,dz\leqslant c\int_{Q_{2R,\varepsilon}} |\nabla u|^2\,dz.$$

Since t_1 is chosen arbitrarily and

$$\int_{B_R} |u(x,t)-\tilde{u}_{R,t}|^2\,dx\leqslant c\int_{B_{2R}} |u(x,t)-\tilde{u}_{2R,t}|^2\,\eta^2\,dx,$$

Lemma 4 follows immediately. ∎

THEOREM 1 (Reverse Hölder Inequality for ∇u). *There exists $p>2$ such that $\nabla u\in L^p_{\mathrm{loc}}(Q)$. Moreover for all $Q_{R,\varepsilon}\subset Q_{16R,\varepsilon}\subset Q$ we have*

$$\left(\fint_{Q_{R,\varepsilon}} |\nabla u|^p\,dz\right)^{1/p}\leqslant c\left(\fint_{Q_{sR,\varepsilon}} |\nabla u|^2\,dz\right)^{1/2}$$

$$+c\left(\fint_{Q_{sR,\varepsilon}} |\nabla\psi|^p\,dz\right)^{1/p}+cR\left(\fint_{Q_{sR,\varepsilon}} |\psi_t|^p\,dz\right)^{1/p}$$

$$+c\sup_{t\in\Lambda_{sR,\varepsilon}}\frac{1}{R}\left(\fint_{B_{sR}} |\psi-\tilde{\psi}_{sR,t}|^p\,dx\right)^{1/p}. \tag{14}$$

Proof. By Lemma 4 and a variant of the Sobolev–Poincaré inequalities as in [9] we have

$$\begin{aligned}
&\int_{Q_{4R,\varepsilon}} |u-\tilde{u}_{4R,t}|^2\,dz \\
&\quad\leqslant \sup_{\Lambda_{4R,\varepsilon}} \left(\int_{B_{4R}} |u(x,t)-\tilde{u}_{4R,t}|^2\,dx\right)^{1/2} \left\{\int_{\Lambda_{4R,\varepsilon}} dt \left(\int_{B_{4R}} |u-\tilde{u}_{4R,t}|^2\,dx\right)^{1/2}\right\} \\
&\quad\leqslant c\left[\int_{Q_{8R,\varepsilon}} |\nabla u|^2\,dz+\Psi_{8R,\varepsilon}\right]^{1/2} \\
&\qquad\cdot\int_{\Lambda_{4R,\varepsilon}} \left[\int_{B_{4R}} |u-\tilde{u}_{4R,t}|^{\gamma}\,dx\right]^{1/2\gamma} \left[\int_{B_{4R}} |u-\tilde{u}_{4R,t}|^{2^*}\,dx\right]^{1/22^*} dt \\
&\quad\leqslant cR^{1/2}\left[\int_{Q_{8R,\varepsilon}} |\nabla u|^2\,dz+\Psi_{8R,\varepsilon}\right]^{1/2} \\
&\qquad\cdot\int_{\Lambda_{4R,\varepsilon}} \left[\int_{B_{4R}} |\nabla u|^{\gamma}\,dx\right]^{1/2\gamma} \left[\int_{B_{4R}} |\nabla u|^2\,dx\right]^{1/4} dt \\
&\quad\leqslant \theta R^2 \int_{Q_{8R,\varepsilon}} |\nabla u|^2\,dx \\
&\qquad+c(\theta)\,R^{-4/N}\left[\int_{Q_{8R,\varepsilon}} |\nabla u|^{\gamma}\,dx\right]^{2/\gamma}+cR^2\Psi_{8R,\varepsilon},
\end{aligned}\tag{15}$$

where $0<\theta\leqslant 1$, $\gamma=2N/(N+2)$, and $2^*=2N/(N-2)$.

From Lemmas 2 and 3 we have

$$\int_{Q_{R,\varepsilon}} |\nabla u|^2\,dz \leqslant c\delta\int_{Q_{4R,\varepsilon}} |\nabla u|^2\,dz+c\Psi_{8R,\varepsilon}+\frac{c}{R^2}\int_{Q_{4R,\varepsilon}} |u-\tilde{u}_{4R,t}|^2\,dz. \tag{16}$$

Combining (15) and (16) we conclude

$$\begin{aligned}
\int_{Q_{R,\varepsilon}} |\nabla u|^2\,dz &\leqslant (c\delta+c\theta)\int_{Q_{8R,\varepsilon}} |\nabla u|^2\,dz \\
&\qquad+c_1R^{-2-4/N}\left[\int_{Q_{8R,\varepsilon}} |\nabla u|^{\gamma}\,dz\right]^{2/\gamma}+c_2\Psi_{8R,\varepsilon},
\end{aligned}\tag{17}$$

where c_1 and c_2 depend on δ and θ. This is enough to prove Theorem 1 since θ and δ can be arbitrarily small (see Proposition 1.1 of Chap. V in [8]). ∎

3. Hölder Continuity of u

In this section we prove that u is Hölder continuous by using a comparison principle as in [3]. We recall the following results for Campanato space due to Da Prato [5].

Lemma 5. *The space $C^{\alpha,\alpha/2}(\bar{Q})$, $\alpha\in(0,1)$, is isomorphic topologically and algebraically to the following space $\mathscr{L}_2^{N+2+2\alpha}(Q)$:*

$$\mathscr{L}_2^{N+2+2\alpha}(Q)=\left\{v\in L^2_{\mathrm{loc}}(Q):\sup_{R,z_0} R^{-N-2-2\alpha}\int_{Q_R}|u-u_{R,z_0}|^2\,dz<\infty\right\}.$$

Then the following lemma is useful for a perturbation argument.

Lemma 6. *Suppose $v\in L^2(-R^2,0;W^{1,2}(B_R))$ is a solution to*

$$v_t-(a_{ij}v_{x_i})_{x_j}=0 \qquad in\ Q_R.$$

Then

$$\int_{Q_\rho}|v-v_\rho|^2\,dz\leqslant c\left(\frac{\rho}{R}\right)^{N+2+\alpha}\int_{Q_R}|v-v_R|^2\,dz, \tag{18}$$

and

$$\int_{Q_\rho}|\nabla v|^2\,dz\leqslant c\left(\frac{\rho}{R}\right)^{N+\alpha}\int_{Q_R}|\nabla v|^2\,dz \tag{19}$$

for all $\rho<R/2$ and for some $\alpha>0$.

We define

$$\Psi_R=\sup_{t\in\Lambda_{R,1}}\int_{B_R}|\psi-\tilde{\psi}_{R,t}|^2\,dx+\int_{Q_R}R^2\,|\psi_t|^2+|\nabla\psi|^2+1\,dz.$$

From Lemma 2 and Poincaré's inequality considering only the x variable we have

$$\int_{Q_\rho}|v-v_\rho|^2\,dz\leqslant c\rho^2\int_{Q_{2\rho}}|\nabla v|^2\,dz. \tag{20}$$

In fact we can find a proof of (20) for constant coefficients $\{a_{ij}\}$ in [2] and for bounded measurable coefficients in [13]. Then (18) follows from Harnack's inequality [12] and (20). We can also prove (19) by Cacciopoli's inequality (8) and Poincaré's inequality (20).

Now we define

$$\Psi_R = \sup_{t \in \Lambda_{R,1}} \int_{B_R} |\psi - \tilde{\psi}_{R,t}|^2 \, dx + \int_{Q_R} R^2 \, |\psi_t|^2 + |\nabla\psi|^2 + 1 \, dz.$$

From Lemma 2 and Poincaré's inequality considering only the x variable we have

$$\int_{Q_\rho} |u - u_\rho|^2 \, dz \leqslant c\rho^2 \int_{Q_{2\rho}} |\nabla u|^2 \, dz + c\rho^2 \Psi_{2\rho}. \tag{21}$$

It is a simple matter to see the following estimates.

LEMMA 7. *Suppose $v \in L^2(-R^2, 0; W^{1,2}(Q_R))$ is a solution to*

$$v_t - (a_{ij} v_{x_j})_{x_i} = \psi_t - (a_{ij} \psi_{x_j})_{x_i}$$

in Q_R. Then

$$\int_{Q_\rho} |v - v_\rho|^2 \, dz \leqslant c \left(\frac{\rho}{R}\right)^{N+2+\alpha} \int_{Q_R} |v - v_R|^2 \, dz + cR^2 \Psi_R$$

and

$$\int_{Q_\rho} |\nabla v|^2 \, dz \leqslant c \left(\frac{\rho}{R}\right)^{N+\alpha} \int_{Q_R} |\nabla v|^2 \, dz + c\Psi_R$$

for all $0 < \rho < R/2$ and for some $\alpha > 0$.

Now we prove Hölder continuity of the solution u to (1).

LEMMA 8. *For each $\rho < R/4$, u satisfies*

$$\begin{aligned} \int_{Q_\rho} &|u - u_\rho|^2 \, dz \\ &\leqslant c \left[\left(\frac{\rho}{R}\right)^{N+2+\alpha} + w\left(\frac{\int_{Q_R} |\nabla u|^2 \, dz + \Psi_R}{R^N}\right) \right] \int_{Q_R} |u - u_R|^2 \, dz \\ &\quad + cR^2 \Psi_R + cR^{N+4} \left[\fint_{Q_R} R^p \, |\psi_t|^p + |\nabla\psi|^p \, dz \right]^{2/p}, \end{aligned} \tag{22}$$

where $w(s) = s^{1-2/p}$ and $p > 2$.

Proof. As in [3], we take $\bar{u} \in L^2(-R^2/4, 0; W^{1,2}(B_{R/2}))$ as the solution of

$$\bar{u}_t - (a_{ij} \bar{u}_{x_j})_{x_i} = \psi_t - (a_{ij} \psi_{x_j})_{x_i}$$
$$\bar{u} = u \text{ on } \partial_p Q_{R/2}.$$

The existence and uniqueness are clear. As usual we get

$$\begin{aligned}\int_{Q_\rho} |u-u_\rho|^2\,dz &\leqslant \int_{Q_\rho} |\bar u-\bar u_\rho|^2\,dz + c\int_{Q_{R/2}} |u-\bar u|^2\,dz\\ &\leqslant c\left(\frac{\rho}{R}\right)^{N+2+\alpha}\int_{Q_{R/2}} |u-u_R|^2\,dz\\ &\quad + c\int_{Q_{R/2}} |u-\bar u|^2\,dz + cR^2\Psi_R \end{aligned}\tag{23}$$

and by Poincaré's inequality

$$\int_{Q_{R/2}} |u-\bar u|^2\,dz \leqslant cR^2\int_{Q_{R/2}} |\nabla u-\nabla\bar u|\,dz. \tag{24}$$

By the maximum principle we know $\bar u(z)\geqslant\psi(z)$ for all $z\in Q_{R/2}$ and $\bar u\in K$. Now we need to estimate the right-hand side of (24). Since $\bar u\in K$, we have from (1)

$$\begin{aligned}&\int_{Q_{R/2}} (\bar u-u)_t\,(\bar u-u)+|\nabla\bar u-\nabla u|^2\,dz\\ &\leqslant c\int_{Q_{R/2}} |\psi_t|\,|\bar u-u|+|\nabla\psi|\,|\nabla\bar u-\nabla u|+(1+|\nabla u|^2)\,|\bar u-u|\,dz. \end{aligned}\tag{25}$$

By Young's inequality and Poincaré's inequality we have

$$\int_{Q_{R/2}} |\psi_t|\,|\bar u-u|\,dz \leqslant c\Psi_R+\varepsilon\int_{Q_{R/2}} |\nabla\bar u-\nabla u|^2\,dz, \tag{26}$$

and

$$\int_{Q_{R/2}} |\nabla\psi|\,|\nabla\bar u-\nabla u|\,dz \leqslant c\Psi_R+\varepsilon\int_{Q_{R/2}} |\nabla\bar u-\nabla u|^2\,dz.$$

By the reverse Hölder inequality

$$\begin{aligned}&\int_{Q_{R/2}} |\nabla u|^2\,|\bar u-u|\,dz\\ &\leqslant c\Bigg[\int_{Q_{3R/4}} |\nabla u|^2\,dz + R^{N+2}\left(\fint_{Q_R} R^p\,|\psi_t|^p+|\nabla\psi|^p\,dz\right)^{2/p}\\ &\quad + \sup_{t\in\Lambda_{R,1}} R^{N+2}\left(\fint_{B_R} |\psi-\tilde\psi_{R,t}|^p\,dx\right)^{2/p}\Bigg]\\ &\quad \cdot\left(\fint_{Q_{R/2}} |\bar u-u|^2\,dz\right)^{1-2/p}. \end{aligned}\tag{27}$$

Combining (23) through (27), we get

$$\begin{aligned}\int_{Q_\rho} |u-u_\rho|^2\,dz \leqslant c\left(\frac{\rho}{R}\right)^{N+2+\alpha} \int_{Q_R} |u-u_R|^2\,dz \\ + cR^2\left(\fint_{Q_{R/2}} |\bar{u}-u|^2\,dz\right)^{1-2/p} \int_{Q_{3R/4}} |\nabla u|^2\,dz + cR^2\Psi_R \\ + cR^{N+4}\left(\fint_{Q_R} R^p\,|\psi_{,t}|^p + |\nabla\psi|^p\,dz\right)^{2/p}. \qquad (28)\end{aligned}$$

Applying Cacciopoli's inequality on the second term of the right-hand side of (28) we have

$$\begin{aligned}\int_{Q_\rho} |u-u_\rho|^2\,dz \leqslant c\left[\left(\frac{\rho}{R}\right)^{N+2+\alpha} + \left(\fint_{Q_{R/2}} |\bar{u}-u|^2\,dz\right)^{1-2/p}\right] \\ \cdot \int_{Q_R} |u-u_R|^2\,dz + cR^2\Psi_R \\ + cR^{N+4}\left(\fint_{Q_R} R^p\,|\psi_{,t}|^p + |\nabla\psi|^p\,dz \right. \\ \left. + \sup \fint_{B_R} |\psi - \tilde{\psi}_{R,t}|^p\,dx\right)^{2/p}. \qquad (29)\end{aligned}$$

Since $\bar{u}$ and u are bounded, we get

$$\int_{Q_{R/2}} |\bar{u}-u|^2\,dz \leqslant cR^2 \int_{Q_{R/2}} |\nabla u|^2\,dz + cR^2\Psi_R. \qquad (30)$$

Combining (29) and (30) we prove Lemma 8. ∎

THEOREM 2. *Suppose* $\nabla\psi \in L^\infty(0, T: L^{q_1}(\Omega))$, $q_1 > N$, *and* $\psi_{,t} \in L^{q_2}(Q)$, $q_2 > (N+2)/2$. *Then* $u \in C^\alpha_{\text{loc}}(Q)$.

Proof. Note that since $\nabla\psi \in L^{q_1}(\Omega)$, $q_1 > N$, with uniform norm with respect to t, we have from Poincaré's inequality and the Sobolev imbedding theorem

$$\sup_{t \in \Lambda_{R,1}} \int_{B_R} |\psi(x,t) - \tilde{\psi}_{R,t}|^2\,dx \leqslant cR^{N+2-2N/q_1}\,\|\nabla\psi\|^2_{L^{q_1}(B_R),\,L^\infty(\Lambda_{R,1})}.$$

Hence by Hölder's inequality we see that

$$R^2\Psi_R \leqslant R^{N+2+2(1-N/q_1)}\,\|\nabla\psi\|^2_{L^{q_1}} + R^{N+2+4(1-(N+2)/2q_2)}\,\|\psi_{,t}\|^2_{L^{q_2}}$$

and

$$R^{N+4}\left(\fint_{Q_R} R^p\,|\psi_t|^p+|\nabla\psi|^p\,dz\right)^{2/p}$$
$$\leqslant cR^{N+2+2(1-N/q_1)}\,\|\nabla\psi\|^2_{L^{q_1},L^\infty}+cR^{N+2+4(1-(N+2)/2q_2)}\,\|\psi_t\|^2_{L^{q_2}}.$$

Now if we show

$$R^{-N}\int_{Q_R}|\nabla u|^2\,dz\to 0\qquad\text{as}\quad R\to 0,\tag{31}$$

then by an iteration lemma [2, 7] we prove Theorem 2. To prove (31), we follow [7]. By Cacciopoli's inequality (8) we have

$$\begin{aligned}R^2\int_{Q_R}|\nabla u|^2\,dz&\leqslant c\int_{Q_{2R}}|u-u_{2R}|^2\,dz+cR^2\Psi_{2R}\\&\leqslant c\int_{Q_{2R}}\max_{Q_{2R}}|\bar u-\bar\psi|^2-|\bar u-\bar\psi|^2\,dz+cR^2\Psi_{2R},\end{aligned}\tag{32}$$

where $\bar u=u-u_{2R}$ and $\bar\psi=\psi-\bar\psi$. Taking $v=u-\rho(\bar u-\bar\psi)\,e^{\mu(\bar u-\bar\psi)^2}\xi$ with $\xi\in C_0^\infty$, $\xi\geqslant 0$, as a competing function to (1) we conclude

$$\begin{aligned}&\int|\nabla\bar u-\nabla\bar\psi|^2\,e^{\mu(\bar u-\bar\psi)^2}\xi\,dz\\&\leqslant\int\{-(\bar u-\bar\psi)^2\}_t\,e^{\mu(\bar u-\bar\psi)^2}\xi\,dz\\&\quad+\int a_{ij}e^{\mu(\bar u-\bar\psi)^2}\{-(\bar u-\bar\psi)^2\}_{x_i}\xi_{x_j}\,dz\\&\quad+c\int e^{\mu(\bar u-\bar\psi)^2}\,|\bar u-\bar\psi|(1+|\nabla\psi|^2+|\psi_t|)\,\xi\,dz\\&\quad+\int a_{ij}(\bar u-\bar\psi)\,e^{\mu(\bar u-\bar\psi)^2}\psi_{x_i}\xi_{x_j}\,dz\end{aligned}$$

for all $\xi\in C_0^\infty$, $\xi\geqslant 0$. We assume that $Q_{4R}(z_0+(0,8R^2))\subset Q$. If we define $w=\max_{Q_{4R}(z_0+(0,8R^2))}|\bar u-\bar\psi|^2-|\bar u-\bar\psi|^2$, then w is a nonnegative supersolution and satisfies

$$\begin{aligned}0&\leqslant\int w_t e^{\mu(\bar u-\bar\psi)^2}\xi\,dz+\int a_{ij}e^{\mu(\bar u-\bar\psi)^2}w_{x_j}\xi_{x_i}\,dz\\&\quad+c\int(1+|\nabla\psi|^2+|\psi_t|)\,\xi\,dz+c\int v_j\xi_{x_j}\,dz\end{aligned}$$

for all $\xi \in C_0^\infty(Q_{4R}(z_0+(0,8R^2)))$, $\xi \geq 0$, where $v_j = a_{ij}(\bar{u}-\bar{\psi})\, e^{\mu(u-\bar{\psi})^2}\psi_{x_j}$. By the time dependent weak Harnack inequality [15] we see

$$\begin{aligned}
&\fint_{Q_{2R}(z_0)} \max_{Q_{4R}(z_0+(0,8R^2))} |\bar{u}-\bar{\psi}|^2 - |\bar{u}-\bar{\psi}|^2\, dz \\
&\quad \leq c\Big(\max_{Q_{4R}(z_0+(0,8R^2))} |\bar{u}-\bar{\psi}|^2 - \max_{Q_R(z_0+(0,8R^2))} |\bar{u}-\bar{\psi}|^2\Big) \\
&\qquad + c\big(R^2 + R^{1-N/q_1} \|\nabla\psi\|^2_{L^{q_1},L^\infty} + R^{2(1-(N+2)/2q_2)} \|\psi_t\|^2_{L^{q_2}}\big). \qquad (33)
\end{aligned}$$

Combining (32) and (33) we have

$$\begin{aligned}
&R^{-N}\int_{Q_R(Z_0)} |\nabla u|^2\, dz \\
&\quad \leq c\Big(\max_{Q_{4R}(z_0+(0,8R^2))} |\bar{u}-\bar{\psi}|^2 - \max_{Q_R(z_0+(0,8R^2))} |\bar{u}-\bar{\psi}|^2\Big) \\
&\qquad + cR^{-N}\Psi_{2R} + c\big(R^2 + R^{1-N/q_1} \|\nabla\psi\|^2_{L^{q_1},L^\infty} + R^{2(1-(N+2)/2q_2)} \|\psi_t\|^2_{L^{q_2}}\big)
\end{aligned}$$

and this is enough to prove (31). ∎

4. Hölder Continuity of ∇u

In this section we assume $a_i(z, u, h)$ satisfies

$$|a_i(z_1, u_1, h) - a_i(z_2, u_2, h)| \leq c(|z_1 - z_2|^{\beta_1} + |u_1 - u_2|^{\beta_2})$$

for some $\beta_1, \beta_2 > 0$ and for all $h \in \mathbb{R}^N$. Also we assume $\nabla\psi \in C^{\beta_3}$, $\beta_3 > 0$, and $\psi_t \in L^q$, $q > N+2$. First we prove the following lemma.

LEMMA 9. *For each $\rho < R/2$, ∇u satisfies*

$$\begin{aligned}
&\int_{Q_\rho} |\nabla u - (\nabla u)_\rho|^2\, dz \\
&\quad \leq c\left(\frac{\rho}{R}\right)^{N+2+\alpha_1} \int_{Q_R} |\nabla u - (\nabla u)_R|^2\, dz + cR^{N+2+\alpha_2} \qquad (34)
\end{aligned}$$

for some $\alpha_1, \alpha_2 > 0$.

Proof. We take $\bar{u} \in L^2(-R^2, 0: W^{1,2}(Q_R))$ as the solution of

$$\begin{gathered}
\bar{u}_t - (a_i(x_0, t_0, (u)_R, \nabla\bar{u}))_{x_i} = \psi_t - (a_i(x_0, t_0, (u)_R, \nabla\psi))_{x_i} \\
\bar{u} = u \qquad \text{on } \partial_p Q_R.
\end{gathered}$$

It is not difficult to see that

$$\begin{aligned}\int_{Q_\rho} |\nabla u-(\nabla u)_\rho|^2\,dz &\leqslant \int_{Q_\rho} |\nabla \bar{u}-(\nabla \bar{u})_\rho|^2\,dz+\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz\\ &\leqslant c\left(\frac{\rho}{R}\right)^{N+2+\alpha_1}\int_{Q_R} |\nabla u-(\nabla u)_R|^2\,dz\\ &\quad + c\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz+cR^{N+2+\alpha_2}\end{aligned}\tag{35}$$

for some $\alpha_1, \alpha_2>0$ (see [11]). As in Lemma 8 we see that

$$\begin{aligned}&\sup_t \int_{B_R} |\bar{u}-u|^2\,dx+\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz\\ &\quad\leqslant c\int_{Q_R} |\psi_t|\,|\bar{u}-u|+|\nabla\psi-(\nabla\psi)_R|\,|\nabla \bar{u}-\nabla u|\,dz\\ &\qquad+\int_{Q_R} (|z-z_0|^{\beta_1}+|u-u_R|^{\beta_2})\,|\nabla \bar{u}-\nabla u|+(1+|\nabla u|^2)\,|\bar{u}-u|\,dz.\end{aligned}\tag{36}$$

We know already that u is Hölder continuous and $|u-u_R|\leqslant cR^{\alpha_3}$ for all $0<\alpha_3<1$. So we have

$$\begin{aligned}\int_{Q_R} (|z-z_0|^{\beta_1}+|u-u_R|^{\beta_2})\,|\nabla \bar{u}-\nabla u|\,dz\\ \leqslant \varepsilon\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz+cR^{N+2+\alpha}\end{aligned}\tag{37}$$

for some $\alpha>0$. By Young's inequality and Poincaré's inequality

$$\begin{aligned}&\int_{Q_R} |\psi_t|\,|\bar{u}-u|\,dz\\ &\quad\leqslant \varepsilon\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz+cR^{N+2+2(1-(N+2)/q)}\,\|\psi_t\|^2_{L^q}.\end{aligned}\tag{38}$$

Since $|\nabla\psi-(\nabla\psi)_R|\leqslant cR^{\alpha}$, we have

$$\begin{aligned}\int_{Q_R} |\nabla\psi-(\nabla\psi)_R|\cdot|\nabla \bar{u}-\nabla u|\,dz\\ \leqslant \varepsilon\int_{Q_R} |\nabla \bar{u}-\nabla u|^2\,dz+cR^{N+2+2\alpha}.\end{aligned}\tag{39}$$

By the reverse Hölder inequality and Poincaré's inequality

$$\begin{aligned}
&\int_{Q_R} |\nabla u|^2 \, |\bar{u} - u| \, dz \\
&\quad \leqslant \left[\int_{Q_R} |\nabla u|^p \, dz \right]^{2/p} \left[\int_{Q_R} |\bar{u} - u|^{p/p-2} \right]^{(p-2)/p} \\
&\quad \leqslant |Q_R|^{2/p-1} \left[\int_{Q_{2R}} |\nabla u|^2 \, dz + R^{N+2} \right] R^{2(p-2)/p} \\
&\qquad \cdot \left[\int_{Q_R} |\nabla \bar{u} - \nabla u|^2 \, dz \right]^{1-2/p}.
\end{aligned} \tag{40}$$

By Cacciopoli's inequality and Hölder continuity of u, we have the following Morrey type growth on ∇u,

$$\int_{Q_{2R}} |\nabla u|^2 \, dz \leqslant c R^{N+\alpha}, \tag{41}$$

for all $0 < \alpha < 1$. If we choose α very close to 1, then we have

$$\int_{Q_R} |\nabla u|^2 \, |\bar{u} - u| \, dz \leqslant \varepsilon \int_{Q_R} |\nabla \bar{u} - \nabla u|^2 \, dz + c R^{N+2+\alpha} \tag{42}$$

for some $\alpha > 0$. Combining (36) through (42) we prove

$$\int_{Q_R} |\nabla \bar{u} - \nabla u|^2 \, dz \leqslant c R^{N+2+\alpha_2} \tag{43}$$

for some $\alpha_2 > 0$. Inequality (34) follows immediately from (35) and (43). ∎

By the iteration lemma, we prove the following theorem.

THEOREM 3. *$\nabla u \in C^{\alpha}_{\mathrm{loc}}$ for some $\alpha > 0$.*

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